

Anomalous wave as a result of the collision of two wave groups on sea surface

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(Dated: February 12, 2016)

The numerical simulation of the nonlinear dynamics of the sea surface has shown that the collision of two groups of relatively low waves with close but noncollinear wave vectors (two or three waves in each group with a steepness of about 0.2) can result in the appearance of an individual anomalous wave whose height is noticeably larger than that in the linear theory. Since such collisions quite often occur on the ocean surface, this scenario of the formation of rogue waves is apparently most typical under natural conditions.

PACS numbers: 47.35.Bb, 92.10.Hm

Anomalous waves (rogue waves, freak waves) constitute one of the most interesting phenomena in the hydrodynamics of the sea surface and create a serious danger to ships. For this reason, they are actively studied (see, e.g., reviews [1-3], special issues of journals [4, 5], and numerous references therein). Among moderate waves, a single very steep wave more than twice as high as neighboring waves sometimes suddenly appears; its height from the trough to the crest reaches 30 m at a characteristic length of an ocean wave of 200-250 m. Such an extreme wave can damage even large ships. Such a large wave exists for several periods and, then, disappears without traces. Several possible mechanisms of this phenomenon have already been proposed. Within the model of potential motions of a liquid in the absence of large-scale inhomogeneous currents, two mechanisms – linear dispersion and nonlinear self-focusing (modulation instability [6, 7]) – are the most remarkable. Let waves be characterized by the following typical parameters: wavenumber k , amplitude \tilde{A} , and the number of waves in a group \tilde{N} . The linear mechanism providing random spatio-temporal focusing is important in wave fields with small Benjamin-Feir indices, $I_{BF} \propto \tilde{N}k\tilde{A} \lesssim 1$ (relatively wide spectrum, short-range spatial correlations, almost complete absence of coherent structures), whereas the nonlinear mechanism is decisive in long-range correlated fields with the presence of coherent structures, where $I_{BF} \gtrsim 1$. It is noteworthy that nonlinearity acts more strongly in long-crested (locally quasi-two-dimensional) than in short-crested (significantly three-dimensional) random wave fields [8-12].

The nonlinear mechanism of rogue waves was studied in numerous works (in addition to the works cited above, see, e.g., [13-16] for planar flows, [17-21] for three-dimensional flows, and references therein). On the contrary, this work concerns the case of small I_{BF} values because it is more typical under natural conditions. However, although the index I_{BF} averaged over all wave groups is small, groups for which $\nu k A \sim 1$ can exist with a certain probability. They play the main role in the formation of anomalous waves.

It is usually accepted that the so-called second-order theory neglecting four-wave interactions describes well the following situation in a field with a wide spectrum. Owing to dispersion, several waves with different lengths and directions can randomly become “in-phase” at a given place and at a given time. Their main harmonics are added according to the linear superposition principle. Nonlinearity only “tunes” higher harmonics (see reviews [1-3] and references therein). However, recent numerical experiments [22, 23] show that nonlinearity is more significantly involved in the formation of a rogue wave even at small I_{BF} values. In particular, nonlinearity elongates its crest and changes the “lifetime”. This work continues the numerical investigation of the effect of nonlinearity on the characteristics of anomalous waves formed through spatio-temporal focusing. The main question that will be answered is as follows. If the initial data are such that dispersion, according to the linear theory, should result not in a single high wave but in the collision of two groups each of two or three waves, can nonlinearity distort a linear interference pattern in the collision time so that a single anomalous wave is formed? A positive answer will be obtained. This result is quite nontrivial. It is important simply because the highest waves in linear fields with a wide spectrum (in particular, in the so-called crossing states when two spectral maxima exist) appear through random collisions of more moderate wave groups, which always appear in the considered region, grow owing to focusing, and then disappear. Events where one high group is directly focused are much rarer. For this reason, it is reasonable to study in detail the pair collision of wave packets on the water surface and the dependence of the properties of appearing anomalous waves on the parameters characterizing the packets and their mutual location at a conditional initial time. This is the aim of the numerical experiments reported below.

In order to better understand the process of collision, it is useful to consider a simple variational model that approximately describes the dynamics of an individual wave packet on deep water. Let x and y be the horizontal coordinates, z be the vertical coordinate, g be the gravitational acceleration, \mathbf{k}_0 be wave vector directed along the x axis, $k_0 = |\mathbf{k}_0|$ be the wavenumber, $\omega_0 = 2\pi/T_0 = \sqrt{gk_0}$ be the frequency of the carrier wave,

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$v_{\text{gr}} = (1/2)\sqrt{g/k_0}$ be the group velocity, and $A(x, y, t)$ be the complex envelope of the main harmonic. The vertical deviation of the free surface is determined by the formula $z \approx \text{Re}[A \exp(i\mathbf{k}_0 \cdot \mathbf{r} - i\omega_0 t)]$. We begin with the corresponding nonlinear Schroedinger equation

$$2i\psi_{\hat{t}} + \psi_{\hat{x}\hat{x}} - \psi_{\hat{y}\hat{y}} + |\psi|^2\psi = 0, \quad (1)$$

written in the dimensionless variables $\psi = k_0 A^*$, $\hat{t} = \omega_0 t$, $\hat{x} = 2k_0(x - v_{\text{gr}}t - x_0)$, and $\hat{y} = \sqrt{2}k_0(y - y_0)$. Substituting the simplest Gaussian ansatz (see, e.g., [23-26] and references therein)

$$\psi = \sqrt{\frac{4N}{XY}} \exp \left[-\frac{\hat{x}^2}{2X^2} - \frac{\hat{y}^2}{2Y^2} + i\frac{U\hat{x}^2}{2X} - i\frac{V\hat{y}^2}{2Y} + i\phi \right] \quad (2)$$

into the Lagrangian of the nonlinear Schroedinger equation

$$\mathcal{L} = \int (i\psi_{\hat{t}}\psi^* - i\psi\psi_{\hat{t}}^* - |\psi_{\hat{x}}|^2 + |\psi_{\hat{y}}|^2 + |\psi|^4/2) d\hat{x}d\hat{y} \quad (3)$$

and performing the standard procedure of the derivation of variational equations, we obtain the homogeneous system

$$\ddot{X} = \frac{1}{X^3} - \frac{N}{X^2Y}, \quad \ddot{Y} = \frac{1}{Y^3} + \frac{N}{Y^2X} \quad (4)$$

for the longitudinal, $X(t)$, and transverse, $Y(t)$, dimensions of the packet. In this case, $4\pi N = \int |\psi|^2 d\hat{x}d\hat{y} = \text{const}$, $U = \dot{X}$, $V = \dot{Y}$. It is worth noting that the parameter N at $Y \sim X$ is proportional to the square of the local Benjamin-Feir index. If $N \sim 1$, dispersion and nonlinear contributions on the right-hand sides are of the same order of magnitude, so that the evolution of a wave packet cannot be divided into the linear and nonlinear stages. The solutions of the system of differential equations (4) are studied in detail [23, 24, 26]. They describe three main stages of the evolution of the wave packet. The stage of focusing corresponds to the ballistic regime with $U \approx \text{const} < 0$ and $V \approx \text{const} < 0$. Then, the stage of the maximum compression of the packet occurs; the details and duration of this stage depend on the initial conditions and can be quite diverse at different N values. Finally, the defocusing stage in the ballistic regime with $U \approx \text{const} > 0$ and $V \approx \text{const} > 0$ occurs. It is worth noting that the applicability of the Gaussian variation ansatz to waves on water even in its more general, off-diagonal variant is confirmed in [23].

In our numerical experiments, two nearly Gaussian packets oriented along the corresponding wave vectors \mathbf{k}_1 and \mathbf{k}_2 (quite close, but noncollinear) at the initial time were centered at the points \mathbf{r}_1 and \mathbf{r}_2 . For simplicity, the parameter N and the initial values X_0 , Y_0 , U_0 , and V_0 were taken to be identical for both packets and corresponded to the ballistic focusing stage. The initial positions of the centers \mathbf{r}_1 and \mathbf{r}_2 were chosen taking into account the group velocities so that collision occurs approximately at the stage of maximum compression. In

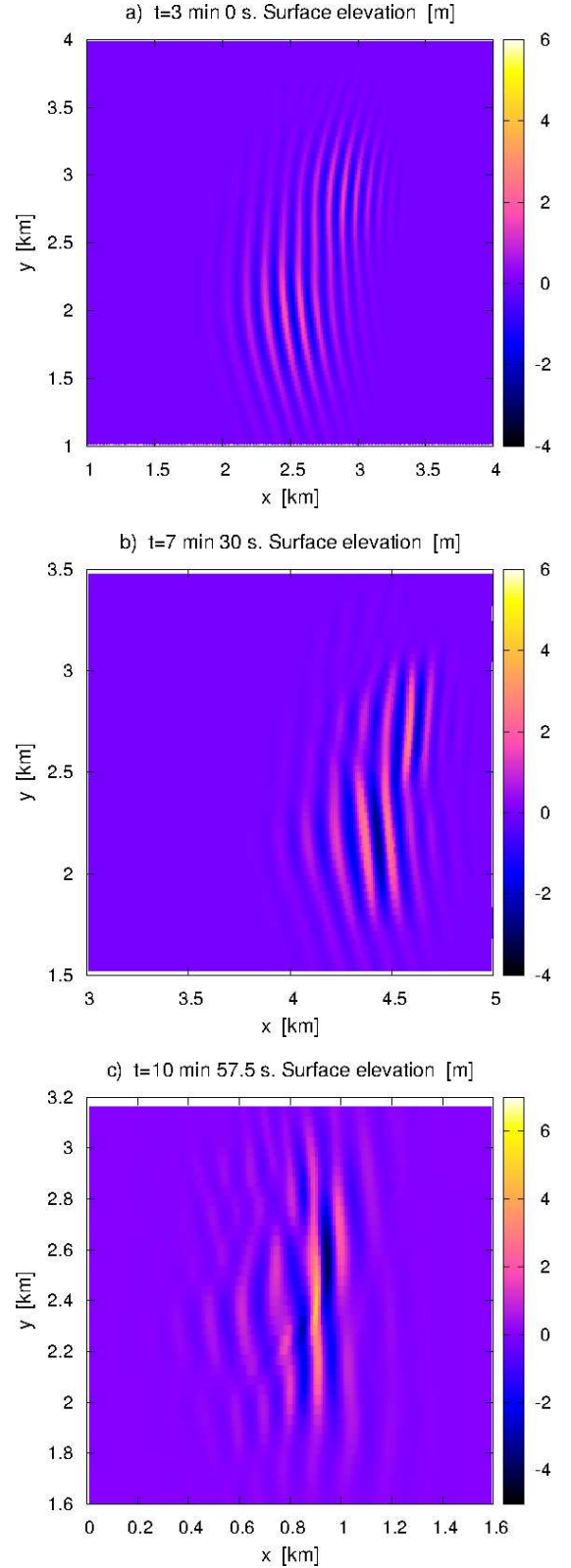


FIG. 1: Successive stages of the collision of two wave packets: (a) focusing, (b) beginning of collision, and (c) appearance of an anomalous wave. The parameters are $N = 1.5$, $X_0 = 30$, $Y_0 = 40$, $U_0 = -0.05$, $V_0 = -0.10$, $\mathbf{k}_1 = (40, 4)$, $\mathbf{k}_2 = (50, -4)$, $\phi_1 = \phi_2 = 0$, $\mathbf{r}_1 = (0.5\pi, 0.75\pi)$, $\mathbf{r}_2 = (0.7\pi, 1.18\pi)$.

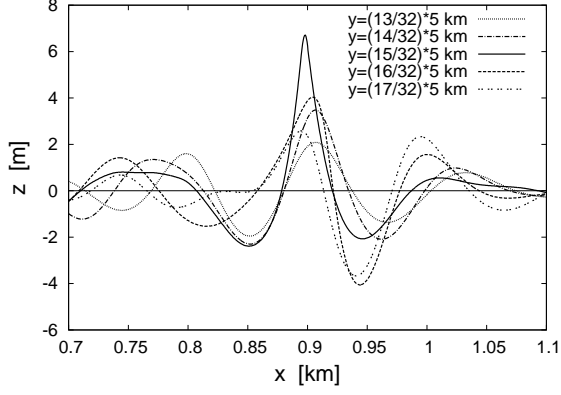


FIG. 2: Several wave profiles from Fig. 1c that confirm the presence of a single anomalous wave. Similar profiles were observed in many numerical experiments with random wave fields (e.g., in [22, 28]), which indicates that the scenario of the formation of rogue waves under consideration is realistic.

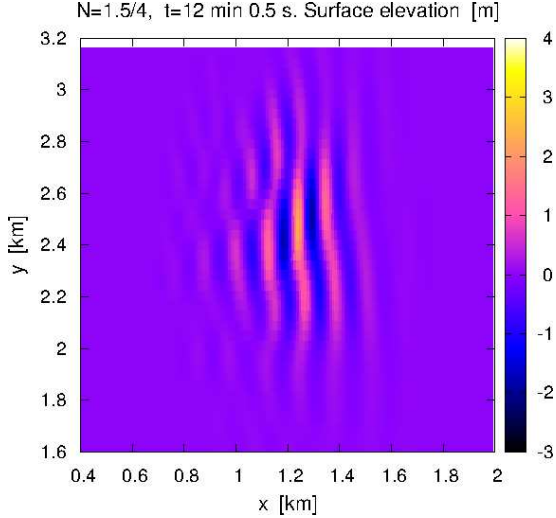


FIG. 3: Wave pattern at the collision of packets with half the initial amplitude of that in Fig. 1. Significant differences from Fig. 1c are caused by an almost linear character of the interaction in this case.

this case, the initial dimensions of the packets should be, on one hand, not too large so that the packets do not significantly overlap and, on the other hand, not too small so that dispersion has no time to defocus them. Despite these constraints, the parametric region of the initial conditions $\mathbf{r}_1 - \mathbf{r}_2$ favorable for the formation of a large wave, which includes states with two maxima, is fairly wide.

The parameter N was chosen such that the steepness of waves at the stage of maximum compression was about 0.2 for each packet. In this case, an extremely nonlinear anomalous wave with a sharp crest was formed at collision. With an increase in N , the wave developed higher, but computations were terminated because of the

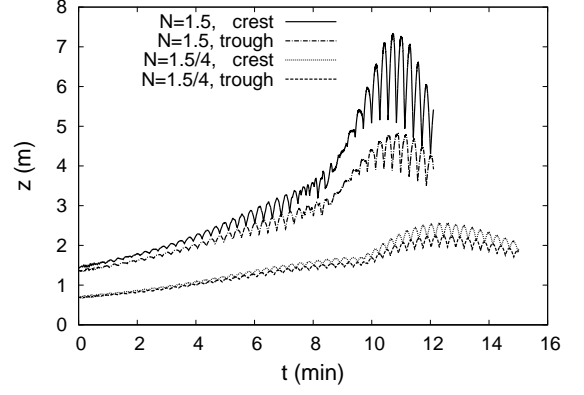


FIG. 4: Time dependence of the height of the highest crest and the depth of the deepest trough at the same parameters as in Fig. 1, as well as similar characteristics at half the initial amplitude. It is seen that the strong nonlinearity led not only to the sharp “top-bottom” asymmetry owing to higher harmonics but also to an about 2.5-fold increase in the average amplitude of the maximum wave (half-sum of envelopes) as compared to the weakly nonlinear regime.

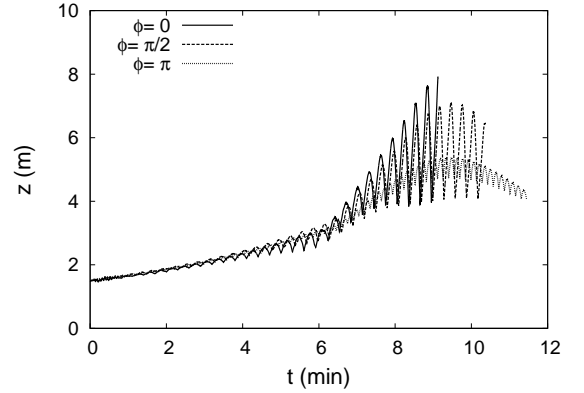


FIG. 5: Time dependence of the height of the highest crest for three ϕ_1 values at $\phi_2 = 0$ and the parameters $N = 1.2$, $X_0 = Y_0 = 30$, $U_0 = V_0 = -0.07$, $\mathbf{k}_1 = (40, 6)$, $\mathbf{k}_2 = (40, -6)$, $\mathbf{r}_1 = (\pi, 0.72\pi)$, $\mathbf{r}_2 = (\pi, 1.28\pi)$.

beginning of the crest breaking, which required overly small spatial and temporal resolutions in our numerical method. At smaller N values, nonlinearity was insufficient.

To simulate the nonlinear dynamics of the free surface, the model of fully nonlinear, weakly three-dimensional water waves described in [27, 28] was used. The computational region was a square with a side of 2π , with the periodic boundary conditions imposed. For better presentation, the results were rescaled to a square with a side of 5 km; as a result, e.g., a dimensionless wavenumber of 50 corresponds to a wavelength of 100 m typical of the World Ocean.

Figure 1 exemplifies the formation of an anomalous

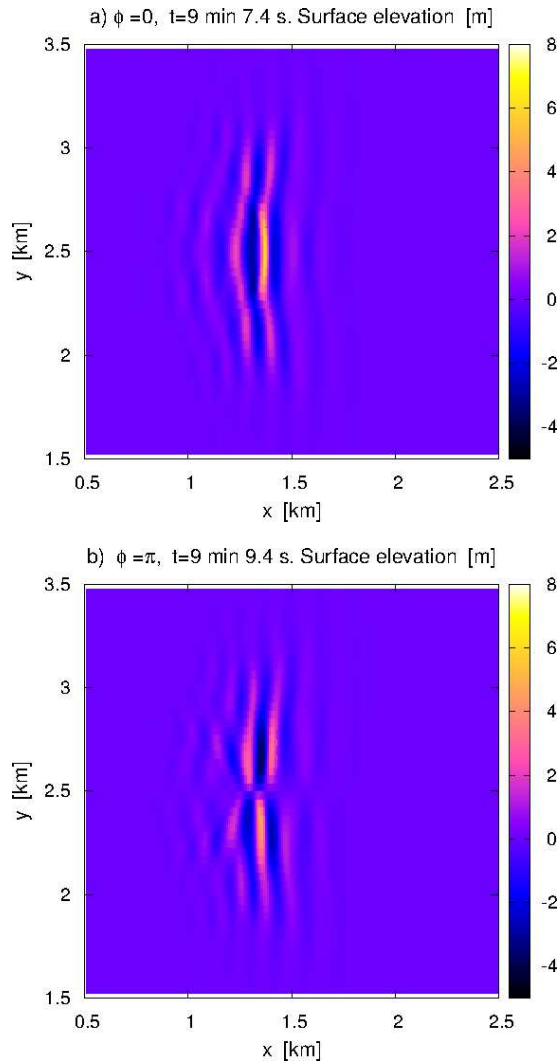


FIG. 6: Example of the effect of the phase difference on the wave pattern at the collision of two wave packets with the same parameters as in Fig. 5.

wave through the collision of two packets. At the first stage, when the packets are separated, their almost independent focusing occurs. Then, the two increased-amplitude regions begin to overlap (beginning of the collision). At this moment, each group consists of two or three waves with a steepness of about 0.2. The length of crests is about three or four wavelengths and the angle between their directions is about 0.2-0.3 rad. Further, the most interesting stage follows, when nonlinearity is sharply enhanced and begins to transform the interfer-

ence pattern from two superimposed wave packets. Instead of a group of two or three waves with the summarized amplitude, which would be observed in the case of the linear superposition of the main harmonics, an extremely short group consisting of nearly one wavelength, i.e., an individual anomalous wave, is formed, as is clearly seen in Figs. 1c and 2 (cf. Fig. 3, where the wave pattern at the collision of packets with half the initial amplitude is shown). It is important that the amplitude of the rogue wave is noticeably larger than the sum of the amplitudes of two groups even if only the main harmonic is taken into account (see Fig. 4). The breather (oscillating) type of this anomalous wave is remarkable: the state with a high crest changes to the state with the deep trough and vice versa, which also follows from Fig. 4. The indicated property is due to the extreme shortness of the group at the double difference between the phase and group velocities. The period of these oscillations is approximately the doubled period of the wave. Undergoing about ten of such oscillations, the large wave expands in the transverse direction (i.e., its crest is elongated), its amplitude decreases, and the wave transfers to the final focusing regime (not shown in the figure).

Calculations with other initial conditions were also performed. In particular, the strong effect of the phase difference $\phi = (\phi_2 - \phi_1)$ on the process of formation of the anomalous wave (under identical other parameters) was revealed. If collision began so that the interference maximum of the linear theory had to pass through the middle of the joined group, the anomalous wave was developed more rapidly and was higher. If the phase difference led to the interference minimum in the middle of the group, the wave was not so high. Such a dependence of the behavior of the solution on the phase difference is exemplified in Figs. 5 and 6.

Collisions of groups at $|\mathbf{k}_1| = |\mathbf{k}_2|$ in our numerical experiments were more efficient than collisions with noticeably different $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$. This indicates that transverse focusing is in some sense more important than longitudinal focusing. In other words, three-dimensionality of the space is fundamentally important for the formation of rogue waves in random fields with low Benjamin-Feir indices.

To summarize, a specific significantly nonlinear mechanism of the formation of anomalous waves at the collision of two quite small and not overly high wave groups at an angle of about 0.2-0.3 rad has been demonstrated. Unfortunately, our model cannot be used to study collisions at larger angles.

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